Well displacing representations and orbit maps

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1   Introduction

1.1   Displacement function

Let γ be an isometry of a metric space X. We recall that the *displacement* of γ is

$$d_X(\gamma) = \inf_{x \in X} d(x, \gamma(x)).$$

We observe that $d_X(\gamma)$ is an invariant of the conjugacy class of γ.

If C is the Cayley graph of a group Γ with set of generators S and word length $\| \ \|_S$, the displacement function is called the *translation length* and is denoted by ℓ

$$\ell_S(\gamma) = \inf_{\eta} \|\eta\gamma\eta^{-1}\|_S.$$

We finally say the action by isometries on X of a group Γ is *well displacing*, if given a set S of generators of Γ, there exist positive constants A and B such that

$$d_X(\gamma) \geqslant A\ell_S(\gamma) - B.$$

This definition does not depends on the choice of S. As first examples, it is easy to check that cocompact groups are well displacing, as well as convex cocompact whenever X is *Hadamard* (i.e. complete, non positively curved and simply connected). We recall that a cocompact action is by definition a properly discontinuous action whose quotient is compact, and a convex cocompact is an action such that there exists a convex invariant on which the action is cocompact.

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The notion naturally arose in [7] where it is shown that for well displacing representations, the energy functional is proper on Teichmüller space and that moreover a large class of representations of surface groups are well displacing.

The purpose of this note is to investigate the relation of well displacing actions, to actions for which the orbit maps are quasi isometric embeddings. For hyperbolic groups the two notions turn out to be equivalent. However in general, the answer is a little more involved.

- There exist infinite groups whose actions are always well displacing. However, an infinite group do have actions for which orbit maps are not quasi isometric embeddings: the trivial action. *Therefore, in general, well displacing representation do not have orbit maps which are quasi isometric embeddings.*

- *There exist a class of group for which if an orbit map is a quasi isometric embedding then the action is well displacing.* The main example is the class of hyperbolic groups. In general these are the groups, for which the stable length is comparable to to the translation length (*cf.* Proposition 3.0.2).

- *There exists a class of group for which every well displacing action have orbit maps which are quasi isometric embeddings.* This class of group are those which are *undistorted in their conjugacy classes* (see Section 4) – or U-property – reminiscent of a statement of Abels, Margulis and Soifer about proximal maps [1]. We prove in Theorem 4.1.1, that some class of linear groups – in particular lattices — enjoys the U-property. We also prove that hyperbolic groups have the U-property.

- *There exists action of groups – which have the U-property – whose orbit maps are quasi isometric embeddings but which are not well displacing.* The simplest example is $\mathrm{SL}(n, \mathbb{Z})$ acting on $\mathrm{SL}(n, \mathbb{R})/SO(n, \mathbb{R})$. More generally in Corollary 5.2.3, we show that if a group has *infinite contortion* (see Section 5.1) – in particular every residually finite group – then any linear representation which contains a unipotent is not well displacing.

Finally, we investigate in a complement a property related to infinite contortion which we call the bounded depth roots property (see Section 5.4). We finish by a simple question : Do all linear groups have U-property?

Contents

2 Infinite groups whose actions are always well displacing

We have

Proposition 2.0.1 *There exists infinite finitely generated groups whose actions are always well displacing. Hence there exists action for which the orbit maps are not quasi isometric embeddings, but which are well displacing.*

PROOF : Denis Osin [9] has constructed infinite finitely generated groups with exactly n conjugacy classes. Any action of such a group is well displacing. For the second part, we just take the trivial action on a point.

3 Orbit maps which are quasi isometry implies well displacing

If a group Γ acts by isometries on a metric space X, we define the *stable norm with respect to X* by

$$[g]_\infty^X = \liminf_{n \to \infty} \frac{1}{n} d(x_0, g^n(x_0)).$$

We observe that this quantity does not depends on the choice of the base point x. The *stable norm* $[g]_\infty$ is the stable norm with respect to the Cayley graph of Γ.

     We now prove the following easy result

Proposition 3.0.2 *If a group Γ is such that there exists $\alpha > 0$ such that*

$$\forall g \in \Gamma, \quad [g]_\infty \geqslant \alpha.\ell(g),$$

then every action of Γ on (X,d) for which the orbit map is a quasi isometric embedding is well displacing.

PROOF : by definition, if the orbit map is a quasi isometric embedding, for every $x \in X$ there exists some constant A and B such that

$$A\|\gamma\| + B \geqslant d(x, \gamma(x)) \geqslant A^{-1}\|\gamma\| - B.$$

It follows that

$$A[\gamma]_\infty \geqslant [\gamma]_\infty^X \geqslant A^{-1}[\gamma]_\infty.$$

Now we remark that

$$[\gamma]_\infty^X \leqslant d_X(\gamma).$$

The result follows.Q.E.D.

REMARKS:

- We will show later on that this inequality fails for $SL(n, \mathbb{Z})$ for $n \geqslant 3$.

- On the other hand, we observe following [4, p 119], that for Γ a hyperbolic group, the stable norm of an element coincides up to a constant with its translation length : there exists a contant K such that $|\ell(g) - [g]_\infty| \leqslant K$.

Therefore, we have

Corollary 3.0.3 *Let Γ be a hyperbolic group. If an isometric action is such that the orbit maps are quasiisometries, then this action is well displacing.*

4 Groups whose well displacing actions have orbit maps which are quasi isometric embeddings

We say a finitely generated group is *undistorted in its conjugacy classes* – in short, has *U-property* – , if there exists finitely many elements $g_1, \ldots g_p$ of Γ, positive constants A and B such that

$$\forall \gamma \in \Gamma, \|\gamma\| \leqslant A \sup_{1 \leqslant i \leqslant n} \ell(g_i \gamma) + B.$$

REMARKS:

- This property is clearly independent of S.

- This property is satisfied by free groups and commutative groups. On the other hands the groups constructed by D. Osin [9] described in the above paragraph do not have this *U*-property.

- Note also that this property is very similar to the statement of Abels, Margulis and Soifer's result [1, Theorem 4.1] and it is no surprise that their result plays a role in the proof of Theorem 4.1.1.

- Finally, by the conjugacy invariance of the translation length, $\ell(g_i\gamma) = \ell(\gamma g_i)$ so we will indifferently write this property with left or right multiplication by the finite family (g_i).

Lemma 4.0.4 *If Γ has U-property, then every well displacing action have orbit maps which are quasi isometric embeddings.*

PROOF : Indeed, assume that Γ acts on X by isometries and that the action is well displacing. In particular we have positive constants α and β so that

$$\forall x \in X, \ \ \gamma \in \Gamma, \ \ d(x, \gamma(x)) \geqslant \alpha \ell_S(\gamma) - \beta.$$

Moreover there exists finitely many elements $g_1, \dots g_p$ of Γ, positive constants A and B such that

$$\forall \gamma \in \Gamma, A\|\gamma\| - B \leqslant \sup_{1 \leqslant i \leqslant n} \ell(g_i\gamma).$$

Hence, let $x \in X$, then

$$
\begin{aligned}
d(x, \gamma(x)) &\geqslant \sup_{1 \leqslant i \leqslant n} d(x, g_i.\gamma(x)) - \sup_{1 \leqslant i \leqslant n} d(x, g_i(x)) \\
&\geqslant \alpha \sup_{1 \leqslant i \leqslant n} \ell(g_i\gamma) - \beta - \sup_{1 \leqslant i \leqslant n} d(x, g_i(x)) \\
&\geqslant \alpha.A\|\gamma\| - B\alpha - \beta - \sup_{1 \leqslant i \leqslant n} d(x, g_i(x)).
\end{aligned}
$$

Hence the orbit map is a quasi isometric embedding. Q.E.D.

We will prove in the next section,

Theorem 4.0.5 *Every uniform lattice – and non uniform lattice in higher rank – in characteristic zero has U-property. In particular every surface group has U-property.*

Moreover, we show that hyperbolic groups have U-property.

4.1 Linear Groups having U-property

We prove the following result which implies Theorem 4.0.5

Theorem 4.1.1 *Let Γ be a finitely generated group and $\mathbf{G}$ a reductive group defined over a field F, suppose that*

- *there exists a homomorphism $\Gamma \to \mathbf{G}(F)$ with Zariski dense image,*

- *there are a finitely many field homomorphisms $(i_v)_{v \in S}$ of F in local fields F_v such that the diagonal embedding $\Gamma \to \Pi_{v \in S}\mathbf{G}(F_v)$ is a quasi-isometric embedding.*

Then the group Γ has U-property.

Since lattices are Zariski dense (Borel theorem) and that higher rank irreducible lattices in characteristic zero are quasi isometrically embedded ([8]), this theorem implies Theorem 4.0.5 for higher rank lattices. The same holds for all uniform lattices. A specific corollary is the following

Corollary 4.1.2 *Let Γ be a finitely generated that is quasi isometrically embedded and Zariski dense in a reductive group $G(F)$ where F is a local field. Then Γ has U-property.*

4.1.1 Generalities for U-property

We prove two lemmas for the U-property.

Lemma 4.1.3 . *Let Γ be a finitely generated group. Let $\Gamma_0 \lhd \Gamma$ be a normal subgroup of finite index. If Γ_0 has U-property, so has Γ.*

We do not know whether the converse statement holds, in other words whether U-property is a property of commensurability classes.

PROOF : We first observe that every finite index subgroup of a finitely generated group is finitely generated. Let S_0 be a generating set for Γ_0 and write Γ as the union of left cosets for Γ_0

$$\Gamma = \bigcup_{t \in T} \Gamma_0 \cdot t.$$

We assume that T is symmetric. Clearly $S = S_0 \cup T$ is a generating set for Γ.

We denote $\| \cdot \|_{\Gamma_0}$ and ℓ_{Γ_0} the word and translation lengths for Γ_0.

We observe that Γ_0 is quasi-isometrically embedded in Γ. Hence there exist positive constants α and β such that

$$\forall \gamma \in \Gamma_0, \quad \|\gamma\|_{\Gamma_0} \geqslant \|\gamma\|_{\Gamma} \geqslant \alpha \|\gamma\|_{\Gamma_0} - \beta. \tag{1}$$

For any γ in Γ, we write $\gamma = \gamma_0 t_0$ with $t_0 \in T$ and $\gamma_0 \in \Gamma_0$. Hence

$$\|\gamma\|_{\Gamma} \leqslant \|\gamma_0\|_{\Gamma_0} + 1 \leqslant A \sup \ell_{\Gamma_0}(\gamma_0 g_i) + B + 1$$

since Γ_0 has U-property.

Finally we need to compare ℓ_{Γ} and ℓ_{Γ_0}. Let δ in Γ_0, then

$$\begin{aligned}
\ell_{\Gamma}(\delta) &= \inf_{t \in T, \eta \in \Gamma_0} \|t \eta t \delta \eta^{-1} t^{-1}\|_{\Gamma} \\
&\geqslant \inf_{\eta \in \Gamma_0} \|\eta \delta \eta^{-1}\|_{\Gamma} - 2 \\
&\geqslant \alpha \inf_{\eta \in \Gamma_0} \|\eta \delta \eta^{-1}\|_{\Gamma_0} - \beta - 2 \\
&\geqslant \alpha \ell_{\Gamma_0}(\delta) - \beta - 2
\end{aligned} \tag{2}$$

Finally combining Inequalities (4.1.1) and (2) , we have

$$\|\gamma\|_{\Gamma} \leqslant \frac{A}{\alpha} \sup_{t \in T, i \in I} \ell_{\Gamma}(\gamma t^{-1} \beta_i) + B + 1 + \frac{\beta + 2}{\alpha}.$$

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This is exactly the U-property for Γ. Q.e.d.

Also

Lemma 4.1.4 *Let Γ be a finitely generated group. Suppose that $\Gamma \to \Gamma_0$ is onto with finite kernel. Then the group Γ has U-property if and only if Γ_0 has.*

Proof : We choose a generating set S for Γ which contains the kernel of $\Gamma \to \Gamma_0$. We choose the generating set S_0 for Γ_0 to be the image of S. Then we have, using surjectivity, for all γ projecting to γ_0

$$\begin{aligned}
\|\gamma\|_\Gamma &\geqslant \quad \|\gamma_0\|_{\Gamma_0} \quad \geqslant \|\gamma\|_\Gamma - 1, \\
\ell_\Gamma(\gamma) &\geqslant \quad \ell_{\Gamma_0}(\gamma_0) \quad \geqslant \ell_\Gamma(\gamma) - 1.
\end{aligned}$$

These two inequalities enable us to transfer the U-property from Γ to Γ_0 and vice versa. Q.e.d.

4.1.2 Proximality

We recall the notion of proximality and a result of Abels, Margulis and Soifer.

Let k be a local field. Let V be a finite dimensional k-vector space equipped with a norm. Let d be the induced metric on $\mathbb{P}(V)$. Let r and ϵ be positive numbers such that
$$r > 2\epsilon.$$

An element g of $\mathrm{SL}(V)$ is said to be (r, ε)-*proximal*, if there exist a point x_+ in $\mathbb{P}(V)$ and an hyperplane H in V such that

- $d(x_+, \mathbb{P}(H)) \geqslant r$,

- $\forall x \in \mathbb{P}(V), \quad d(x, \mathbb{P}(H)) \geqslant \varepsilon \implies d(g \cdot x, x_+) \leqslant \varepsilon$.

In particular a proximal element has a unique eigenvalue of highest norm. Conversely, if an element g admits a unique eigenvalue of highest norm, then some power of g is proximal (for some (r, ϵ)).

We cite the needed result from [1] and [2].

Theorem 4.1.5 ([1] Theorem 5.17) *Let $\mathbf{G}$ a semisimple group over a field F. Let $(i_v)_{v \in V}$ be finitely many field homomorphisms of F in local fields F_v. Let $\rho_v : \mathbf{G}(F_v) \to \mathrm{GL}(n_v, F_v)$ be an irreducible representation of $\mathbf{G}(F_v)$ for each v.*

Suppose that Γ is a Zariski dense subgroup of $\mathbf{G}(F)$. Suppose that for every v, $\rho_v(\Gamma)$ contains proximal elements. Then there exist

- *$r > 2\varepsilon > 0$*

- *a finite subset $\Delta \subset \Gamma$,*

such that for every γ in Γ there is some δ in Δ such that $\rho_v(\gamma s)$ is (r, ε)-proximal for every v in V.

This result is usually stated with *one* local field but the proof of the above extension and the following is straightforward.

We shall also need the following Lemma

Lemma 4.1.6 ([2] Corollaire p.13) *Let Γ be a Zariski dense subgroup in $\mathbf{G}(k)$, $\mathbf{G}$ a reductive group over a local field k. Then Γ is unbounded if and only if there exists an irreducible representation of $\mathbf{G}(k)$, such that $\rho(\Gamma)$ contains a proximal element.*

The special case of $k = \mathbb{R}$ was proved in [3].

4.1.3   Proximal elements and translation lengths

We recall some facts on length and translation length in $\mathbf{G}(k)$ where $\mathbf{G}$ is a semisimple group over a local field k.

Let K be the maximal compact subgroup of $\mathbf{G}(k)$. This defines a norm $\|g\|_G = d_{G/K}(K, gK)$ in $\mathbf{G}(k)$ which satisfies

$$\|gh\|_G \leqslant \|g\|_G + \|h\|_G.$$

We also consider the translation length ℓ_G in $\mathbf{G}(k)$:

$$\ell_G(g) = \inf_{h \in G} \|hgh^{-1}\|_G.$$

Observe that the translation norm is actually independent of the choice of the maximal compact subgroup since they are all conjugated. The translation length and norm of (r, ϵ)-proximal elements can be compared :

Lemma 4.1.7 (compare [2]§ 4.5) *Let $\mathbf{G}$ be a semisimple group over a local field k. Let $\rho : \mathbf{G}(k) \to \mathrm{GL}(n, k)$ be an irreducible representation. Let $\varepsilon > 0$.*

Then there exist positive constants α and β such that if $\rho(g)$ is (r, ε)-proximal then

$$\ell_G(g) \geqslant \alpha \|g\|_G - \beta \tag{3}$$

PROOF : We use classical notation and refer to [2, p.6–8] for precise definitions.

Any g in $\mathbf{G}(k)$ is contains in a unique double coset $K\mu(g)K$. The element $\mu(g) \in A^+ \subset A$ is called the Cartan projection of g. Here we see A^+ as a subset of a cone $A^\times$ in some $\mathbf{R}$-*vector space*.

For some integer n ($n = 1$ in the archimedean case) the element g^n admits a Jordan decomposition $g^n = g_e g_h g_u$ with g_e, g_h, g_u commuting, g_e elliptic, g_u unipotent and g_h hyperbolic, *i.e.* conjugated to a unique element $a \in A^+$. We set $\lambda(g) = \frac{1}{n} a \in A^\times$.

If we fix some norm on the vector space containing the cone $A^\times$ then (up to quasi isometry constants) the norm of $\mu(g)$ is $\|g\|_G$ in $\mathbf{G}(k)$ and the norm of $\lambda(g)$ is $\ell_G(g)$.

Then by a result of Y. Benoist [2], there exists a compact subset N_ε of the vector space containing $A^\times$ such that for every g such that $\rho(g)$ is (r, ε)-proximal we have

$$\lambda(g) - \mu(g) \in N_\varepsilon.$$

The lemma follows. Q.E.D.

4.1.4   Proof of Theorem 4.1.1

By taking a finite index normal subgroup and projecting (Lemmas 4.1.3 and 4.1.4) we can make the hypothesis that $\mathbf{G}$ is the product of a semisimple group $\mathbf{S}$ and a torus $\mathbf{T}$ and Γ is a subgroup of $\mathbf{S}(F) \times \mathbf{T}(F)$.

Moreover, since length and translation length for elements in $\mathbf{T}(F_v)$ are equal, we only need to work with the semisimple part $\mathbf{S}$.

Finally, it suffices to prove the existence of a finite family $F \subset \Gamma$ and constants A, B such that, for any $\gamma \in \Gamma$

$$\|\gamma\|_S \leqslant A \sup_{f \in F} \ell_S(\gamma f) + B, \tag{4}$$

where $S = \Pi_{v \in V} \mathbf{S}(F_v)$. Indeed, since Γ is quasiisometrically embedded in G, $\|\gamma\|_\Gamma$ is less than $\|\gamma\|_G = \|\gamma\|_T + \|\gamma\|_S$ and $\ell_G(\gamma) = \ell_T(\gamma) + \ell_S(\gamma) = \|\gamma\|_T + \ell_S(\gamma)$ is less than $\ell_\Gamma(\gamma)$ –up to quasi isometry constants– and Inequality 4 implies that Γ has U-property.

Note that we can forget any completion F_v where the subgroup $\Gamma \subset \mathbf{S}(F_v)$ is bounded without changing the fact that Γ is quasiisometrically embedded. So by Lemma 4.1.6, for each v there is an irreducible representation $\rho_v : \mathbf{S}(F_v) \to \mathrm{GL}(n_v, F_v)$ such that $\rho_v(\Gamma)$ contains proximal elements.

Applying Theorem 4.1.5 we find a finite family $F \subset \Gamma$ and r, ε such that for every γ in Γ there is some $f \in F$ such $\rho_v(\gamma f)$ is (r, ε)-proximal for every v. Hence, as consequence of Lemma 4.1.7, we have for such γ and f :

$$\|\gamma f\|_S \leqslant A \ell_S(\gamma f) + B.$$

This implies Inequality (4) and concludes the proof.

4.2   Hyperbolicity and the U-property

Let Γ be a finitely generated group and S a set of generators. Let d be its word distance and $\|g\| = d(e, g)$. We denote by

$$\langle g, h \rangle_u = \frac{1}{2}(d(g, u) + d(h, u) - d(g, h)),$$

the *Gromov product* – based at u – on Γ. We abbreviate $\langle g, h \rangle_e$ by $\langle g, h \rangle$. Observe that

$$\langle gu, hu \rangle_u = \langle g, h \rangle_e. \tag{5}$$

Recall that Γ is called δ-*hyperbolic* if for all g, h, k in Γ we have

$$\langle g, k \rangle \geqslant \inf(\langle g, h \rangle; \langle h, k \rangle) - \delta \tag{6}$$

and Γ is called *hyperbolic* if it is δ-hyperbolic for some δ. A hyperbolic group is called *non elementary* if it is not finite and does not contain $\mathbb{Z}$ as a subgroup of finite index.

Then

Proposition 4.2.1 *Hyperbolic groups have U-property.*

We recall the *stable translation length of an element g*:

$$[g]_\infty = \lim_{n\to\infty} \frac{\|g^n\|}{n}.$$

We remark that obvioulsy

$$[g]_\infty \leqslant \ell(g).$$

We shall actually prove

Proposition 4.2.2 *Let Γ be hyperbolic. There exist a pair $u, v \in \Gamma$ and a constant α such that for every g one has*

$$\|g\| \leqslant 3\sup([g]_\infty, [gu]_\infty, [gv]_\infty) + \alpha.$$

In particular Γ has property U.

REMARK:

- Let Γ be a free group generated by some elements $u, v, w_1, \ldots w_n$. Then G is 0 hyperbolic. If $[g] \neq \|g\|$ the first letter of g must be equal to the inverse of the last one. Multipliying either by u or by v we find a new element which is cyclically reduced : for this element the stable translation length and the length are equal. The proof of Proposition 4.2.2 is a generalisation of this remark.

4.2.1 Almost cyclically reduced elements

An element g in Γ is said to be *almost cyclically reduced* if $\langle g, g^{-1}\rangle \leqslant \frac{\|g\|}{3} - \delta$ We prove in this paragraph

Lemma 4.2.3 *If g is almost cyclically reduced , then*

$$[g]_\infty \geqslant \frac{\|g\|}{3}.$$

The following result [5, Lemme 1.1] will be useful.

Lemma 4.2.4 *Let (x_n) be a finite or infinite sequence in G. Suppose that*

$$d(x_{n+2}, x_n) \geqslant \sup(d(x_{n+2}, x_{n+1}), d(x_{n+1}, x_n)) + a + 2\delta,$$

or equivalently that

$$\langle x_{n+2}, x_n\rangle_{x_{n+1}} \leqslant \frac{1}{2}\inf(d(x_{n+2}, x_{n+1}), d(x_{n+1}, x_n)) - \frac{a}{2} - \delta.$$

Then

$$d(x_n, x_p) \geqslant |n - p|a.$$

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This implies Lemma 4.2.3.

PROOF : Let $x_n = g^n$. By left invariance and since g is almost cyclically reduced,

$$\langle x_{n+2}, x_n \rangle_{x_{n+1}} = \langle g, g^{-1} \rangle \leqslant \frac{\|g\|}{2} - \frac{a}{2} - \delta,$$

for $a = \frac{\|g\|}{3}$. By Lemma 4.2.4,

$$\|g^n\| \geqslant n \frac{\|g\|}{3}.$$

The result follows Q.E.D.

4.2.2 Ping pong pairs

A *ping pong pair* in Γ is a pair of elements u, v such that :

1. $\inf(\|u\|, \|v\|) \geqslant 100\delta$.

2. $\langle u^{\pm 1}, v^{\pm 1} \rangle \leqslant \frac{1}{2} \inf(\|u\|, \|v\|) - 20\delta$

3. $\langle u, u^{-1} \rangle \leqslant \frac{\|u\|}{2} - 20\delta$ and $\langle v, v^{-1} \rangle \leqslant \frac{\|v\|}{2} - 20\delta$

REMARKS:

- A ping pong pair generates a free subgroup. This is an observation from [5]. To prove this, consider a reduced word w on the letter u, v, u^{-1}, v^{-1}. If x_n is the prefix of length n of w, the sequence x_n satisfies the hypothesis of Lemma 4.2.4.

- In the present proof, the third property will not be used.

We shall prove

Lemma 4.2.5 *If Γ is hyperbolic non-elementary, there exists a ping pong pair.*

PROOF : In [6] explicit ping pong pairs are constructed. Here is a construction whose idea goes back to F. Klein. Let f be some hyperbolic element (an element of infinite order). Replacing f by a conjugate of some power, we may assume that

$$\|f\| = [f] > 1000\delta.$$

As Γ is not elementary, there exists a generator a of Γ which do not fix the pair of fixed points f^+, f^- of f on the boundary $\partial\Gamma$: otherwise, since the action of Γ is topologically transitive, $\partial\Gamma$ would be reduced to these two points and Γ would be elementary. Now, let us prove that for some integer N, $(f, af^N a^{-1})$ is a ping pong pair. We have

$$\lim_{N \to +\infty} f^N = f^+ \neq af^+ = \lim_{N \to +\infty} af^N a^{-1}.$$

It follows that the Gromov product $\langle f^N, af^N a^{-1} \rangle$ remains bounded, by the very definition of the boundary. Hence, for N large enough, we have

$$(f, af^N a^{-1}) \leqslant \frac{1}{2} \inf(\|f^N\|, \|af^N a^{-1}\|) - 20\delta.$$

A similar argument also yields that $\langle f^N, af^{-N} a^{-1} \rangle$ remains bounded. Therefore $(f^N, af^N a^{-1})$ is a ping pong pair for $N \gg 1$. Q.E.D.

4.2.3 Proof of Proposition 4.2.2

We first reduce this proof to the following Lemma

Lemma 4.2.6 *Let (u, v) be a ping pong pair. Let $g \in \Gamma$ such that*

$$\|g\| \geqslant 3 \sup(\|u\|, \|v\|) + 100\delta.$$

Then one of the three elements g, gu, gv is almost cyclically reduced.

We oberve at once that Proposition 4.2.2 follows from Lemma 4.2.3, 4.2.5 and 4.2.6 : choose a ping pong pair u, v and take

$$\alpha = 3 \sup(\|u\|, \|v\|) + 100\delta.$$

PROOF : Assume g is not almost cyclically reduced. Then

$$\langle g, g^{-1} \rangle \geqslant \frac{\|g\|}{3} - \delta \geqslant \sup(\|u\|, \|v\|) + 30\delta. \tag{7}$$

Moreover one of the following pair of inequalities hold

$$\langle g^{-1}, u^{\pm 1} \rangle \leqslant \frac{\|u\|}{2} - 10\delta, \tag{8}$$

or

$$\langle g^{-1}, v^{\pm 1} \rangle \leqslant \frac{\|v\|}{2} - 10\delta, \tag{9}$$

Otherwise, by the definition of hyperbolicity we would have for some $\varepsilon, \varepsilon' \in \{\pm 1\}$,

$$\langle u^\varepsilon, v^{\varepsilon'} \rangle \geqslant \frac{1}{2} \inf(\|u\|, \|v\|) - 10\delta - \delta,$$

contradicting the second property of the definition of ping pong pairs.

So we may assume that Inequality 8 holds. We will show that gu is almost cyclically reduced. Let $k = gu$. By the triangle inequality $\langle u^{-1}, g \rangle \leqslant \|u\|$. Thus from Inequality 7, we deduce that

$$\inf(\langle g, u^{-1} \rangle, \langle g, g^{-1} \rangle) = \langle g, u^{-1} \rangle.$$

Then, by the definition of hyperbolicity, we get:

$$\langle g, u^{-1} \rangle \leqslant \langle u^{-1}, g^{-1} \rangle + \delta. \tag{10}$$

Using Inequality 8 now, we have:

$$\langle g, u^{-1} \rangle \leqslant \frac{\|u\|}{2} - 9\delta. \tag{11}$$

Note that

$$\langle g, k \rangle = \frac{1}{2}(\|g\| + \|gu\| - \|u\|) \geqslant \frac{1}{2}(2\|g\| - 2\|u\|) \geqslant 2\|u\| + 100\delta.$$

By the triangle inequality again

$$\langle k, u^{-1} \rangle \leqslant \|u\|.$$

Therefore

$$\inf(\langle g, k \rangle, \langle k, u^{-1} \rangle) = \langle k, u^{-1} \rangle.$$

From hyperbolicity and Inequality 11, we have

$$\langle k, u^{-1} \rangle \leqslant \langle g, u^{-1} \rangle + \delta \leqslant \frac{\|u\|}{2} - 8\delta \tag{12}$$

Applying successively Inequalities 8 and 12, we get that

$$\langle k^{-1}, u^{-1} \rangle = \|u\| - \langle u, g^{-1} \rangle \geqslant \frac{\|u\|}{2} + 10\delta \geqslant \langle k, u^{-1} \rangle + 18\delta. \tag{13}$$

By hyperbolicity,

$$\inf(\langle k, k^{-1} \rangle, \langle k^{-1}, u^{-1} \rangle) \leqslant \langle k, u^{-1} \rangle + \delta.$$

Therefore, Inequalities 12 and 13 imply that

$$\langle k, k^{-1} \rangle \leqslant \langle k, u^{-1} \rangle + \delta \leqslant \frac{\|u\|}{2} - 7\delta \tag{14}$$

Since $\|k\| \geqslant \|g\| - \|u\| \geqslant 3\|u\| - \|u\| + 100\delta$, we finally obtain that k is almost cyclically reduced. Q.E.D.

5 Non well displacing actions whose orbit maps are quasi isometric embeddings

We prove in particular

Proposition 5.0.7 *The action of* $\mathrm{SL}(n, \mathbb{Z})$ *on* $X_n = \mathrm{SL}(n, \mathbb{R})/SO(n, \mathbb{R})$ *is not well displacing, although, for* $n \geqslant 3$*, the orbit maps are quasi isometric embeddings.*

The second part of this statement is a theorem of Lubotzky, Mozes and Ragunathan [8]. Note that for the action of $\mathrm{SL}(2, \mathbb{Z})$ on the hyperbolic plane $\mathbb{H}_2 = X_2$ the orbit maps are not quasi isometries so it is obviously not well displacing since $\mathrm{SL}(2, \mathbb{Z})$ is a hyperbolic group (see Corollary 3.0.3).

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5.1 Infinite contortion

We say a group Γ has *infinite contortion*, if the set of conjugacy classes of powers of every non torsion element is infinite. In other words, for every non torsion element γ, for every finite family $g_1, \ldots, g_q$ of conjugacy classes of elements of Γ, there exists $k > 0$ such that

$$\forall i \in \{1, \ldots, q\}, \gamma^k \notin g_i.$$

We prove

Lemma 5.1.1 *Every residually finite group has infinite contortion.*

PROOF : Let γ be a non torsion element. Let $g_1, \ldots, g_n$ be finitely many conjugacy classes. We want to prove that there exists $k > 0$ such that γ^k belongs to no g_i. Since γ is not a torsion element, we can assume that all the g_i are non trivial. Let $h_i \in g_i$. Since all h_i are non trivial, by residual finiteness there exist a homomorphism ϕ in a finite groups H, such that

$$\forall i, \varphi(h_i) \neq 1.$$

Let $k = \|H\|$, hence $\varphi(\gamma^k) = 1$. This implies that $\gamma^k \notin g_i$ Q.E.D.

5.2 Displacement function and infinite contortion

We will prove

Lemma 5.2.1 *Assume Γ has infinite contortion. Assume Γ acts cocompactly and properly discontinuoulsy by isometry on a space X. Assume furthermore that every closed bounded set in X is compact. Then, for every non torsion element γ in Γ, we have*

$$\limsup_{p \to \infty} d_X(\gamma^p) = \infty.$$

REMARKS:

- We should notice that the conclusion immediately fails if Γ does not have infinite contortion. Indeed there exists an element γ such that it powers describe only finitely many conjugacy classes of elements $g_1 \ldots g_q$, and hence

$$\limsup_{p \to \infty} d_X(\gamma^p) \leq \sup_{i \in \{1, \ldots, q\}} d_X(g_i) < \infty.$$

- It is also interesting to notice that there are groups with infinite contortion which possess elements γ such that

$$\liminf_{p \to \infty} d_X(\gamma^p) < \infty.$$

Indeed, there are finitely generated linear groups which contain elements γ which are conjugated to infinitely many of its powers. Hence, for such γ we have

$$\liminf_{p \to \infty} d_X(\gamma^p) \leqslant d_X(\gamma).$$

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Here is a simple example. We take $\Gamma = \mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$ and

$$\gamma = \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right).$$

Then for all n, γ^{p^n} is conjugated to γ.

- However, in Paragraph 5.4, we shall give a condition - *bounded depth roots* (satisfied, for example, by any group commensurable to a subgroup of $\mathrm{SL}(n, \mathbb{Z})$) so that together with the hypothesis of previous Lemma

$$\lim_{p \to \infty} d_X(\gamma^p) = \infty.$$

PROOF : We want to prove that

$$\limsup_{p \to \infty} \inf_{x \in X} d(x, \gamma^p x) = \infty.$$

Assume the contrary, then there exists

- a constant R,

- a sequence of points x_i of points in X,

such that for every p,
$$d(x_i, \gamma^p x_i) \leqslant R.$$

Let now K be a compact in X such that $\Gamma.K = X$. Let $f_i \in \Gamma$ such that $y_i = f_i^{-1}(x_i) \in K$. Then
$$d(y_i, f_i^{-1} \gamma^p f_i(y_i)) \leqslant R.$$

Let $K_R = \{z \in X, d(z, K) \leqslant R\}$. It follows that

$$\forall p, \quad f_i^{-1} \gamma^p f_i (K_R) \cap K_R \neq 0.$$

Observe that K_R is compact. By the properness of the action of Γ, we conclude that the family $\{g_i^{-1} \gamma_i^p g_i\}$ is finite. Hence the family of conjugacy classes of the sequence γ^p is finite. But this contradicts infinite contortion for Γ. Q.E.D.

Corollary 5.2.2 *Assume Γ has infinite contortion. Let C be its Cayley graph, then for γ a non torsion element*

$$\limsup_{p \to \infty} \ell(\gamma^p) = \infty.$$

Corollary 5.2.3 *Assume Γ has infinite contortion. Let ρ be a representation of dimension n. Assume $\rho(\Gamma)$ contains a non trivial unipotent, then ρ is not well displacing on $X_n = \mathrm{SL}(n, \mathbb{R})/SO(n, \mathbb{R})$.*

PROOF : Assume ρ is well displacing. Let γ such that $\rho(\gamma)$ is a non trivial unipotent. Then for all p, $d_{X_n}(\gamma^p) = 0$. However γ is not a torsion element. We obtain the contradiction using the previous lemma. Q.E.D.

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5.3 Non uniform lattices

Lemma 5.3.1 *For $n \geqslant 3$, the action of $\mathrm{SL}(n, \mathbb{Z})$ on X_n is such that the orbit maps are quasi isometric embeddings. But it is not well displacing.*

PROOF : The group $\mathrm{SL}(n, \mathbb{Z})$ is residually finite. Hence it has infinite contortion by Lemma 5.1.1. The standard representation ρ contains a non trivial unipotent hence it is not well displacing by Corollary 5.2.3.

By a theorem of Lubotzky, Mozes and Ragunathan [8] irreducible higher rank lattices Λ are quasi-isometrically embedded in the symmetric space. Q.E.D.

5.4 Bounded depth roots

This section is complementary. We say that group Γ has *bounded depth roots property*, if for every γ in Γ which is a non torsion element, there exists some integer p, such that we have

$$q \geqslant p, \eta \in \Gamma \implies \eta^q \neq \gamma.$$

Observe that $\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$ does not have bounded depth roots. Note that this property is well behaved by taking subgroups and is a property of commensurability (see Lemma 5.4.4).

We prove

Proposition 5.4.1 *The following groups have bounded depth roots property*

- *The group $\mathrm{SL}(n, \mathbb{Z})$,*

- *$\mathrm{SL}(n, \mathcal{O})$ where $\mathcal{O}$ is the ring of integers of a number field F,*

- *any subgroup of a group having bounded depth roots property or any group commensurable to a group having this property,*

- *in particular any arithmetic lattice in a archimedian Lie group.*

Lemma 5.4.2 [BOUNDED DEPTH ROOT] *Let Γ be a group with bounded depth roots property. Assume Γ acts cocompactly and properly discontinuously by isometry on a space X. Assume furthermore that every closed bounded set in X is compact. Then, for every non torsion element γ in Γ, we have*

$$\lim_{p \to \infty} d_X(\gamma^p) = \infty.$$

For the proof see Pargaraph 5.4.3. The Lemma and Proposition above again imply that the action of $\mathrm{SL}(n, \mathbb{Z})$ on X_n is not well displacing.

5.4.1 Bounded depth roots property for $\mathrm{SL}(n,\mathbb{Z})$

We prove the above Proposition.

Lemma 5.4.3 $\mathrm{SL}(n,\mathbb{Z})$ *has bounded depth roots.*

PROOF : Let $A \in \mathrm{SL}(n,\mathbb{Z})$. Let $B \in \mathrm{SL}(n,\mathbb{Z})$. We assume there exists k such that $B^k = A$. Let $\{\lambda_j^A\}$ and $\{\lambda_j^B\}$ be the eigenvalues of A and B respectively. Let

$$K = \sup_j |\lambda_j^A|.$$

Then,

$$\sup_j |\lambda_j^B| \leqslant K^{\frac{1}{k}} \leqslant K.$$

Hence all the coefficients of the characteristic polynomial of B have a bound K_1 which only depends on A. Therefore, since these coefficients only take values in $\mathbb{Z}$, it follows the characteristic polynomials of B belongs to the finite family

$$\mathcal{P} = \{P(x) = x^n + \sum_{k=0}^{k=n-1} a_k x^k \ : \ a_k \in \mathbb{Z}, |a_k| \leqslant K_1\}$$

Since $\mathcal{P}$ is finite, there exists a constant $b > 1$ such that for every root λ of a polynomial $P \in \mathcal{P}$,

$$|\lambda| > 1 \implies |\lambda| \geqslant b.$$

Let $q \in N$ be such that b^q is greater than K. It follows that if $B^q = A$, then all eigenvalues of B have complex modulus 1. Therefore the same holds for A.

It follows from this discussion that we can reduce to the case where

$$\forall i,j, |\lambda_j^A| = 1.$$

We say such an element has *trivial hyperbolic part*. Note that necessarily also B has trivial hyperbolic part.

We first prove that there exist a integer M depending only on n, such that if $C \in \mathrm{SL}(n,\mathbb{Z})$ has a trivial hyperbolic part then C^M is unipotent. The same argument as above show that the characteristic polynomials of elements with a trivial hyperbolic part belong a finite family of the form

$$\mathcal{P} = \{P(x) = x^n + \sum_{k=0}^{k=n-1} a_k x^k \ : \ a_k \in \mathbb{Z}, |a_k| \leqslant K_2\}.$$

Where K_2 depends only on n. Note that roots of polynomials belonging to $\mathcal{P}$ that are of complex modulus 1 are roots of unity. Thus we may take M to be a common multiple of the orders of those roots of unity and deduce that C^M is unipotent if $C \in \mathrm{SL}(n,\mathbb{Z})$ has a trivial hyperbolic part.

Returning to our setting we can replace A by A^M and B by B^M and consider $A = B^k$ where both A and B are unipotents. There is some rational matrix $g_0 \in \mathrm{SL}(n,\mathbb{Q})$ depending on A such that $A_0 = g_0 A g_0^{-1}$ is in a Jordan form. We

claim that $B_0 = g_0 B g_0^{-1}$ is a made of blocks which correspond to the Jordan blocks of A_0 and each such block of B_0 is an upper triangular unipotent matrix (This follows from observing that for unipotent matrices a matrix and its powers have the same invariant subspaces). Moreover note that the denominators of the entries of B_0 are bounded by some $L \in \mathbb{N}$ depending only on g_0 (and thus on A). By considering (some of the) entries just above the main diagonal it is easily seen that $k \leq L$. Q.E.D.

5.4.2 Commensurability

We observe

Lemma 5.4.4 *If Γ is commensurable to a subgroup of a group which has bounded depth roots. Then Γ has bounded depth root.*

PROOF : By definition a subgroup of a group having bounded depth roots has bounded depth roots. Let G be a group and H a subgroup having finite index k. Observe that for every element g of G, we have $g^k \in H$. It follows that if H has bounded depth roots, then G has bounded depth roots. Q.E.D.

5.4.3 Proof of Lemma 5.4.2.

PROOF : Let K be a compact in X. We first prove that

$$\lim_{p \to \infty} \inf_{x \in K, \eta \in \Gamma} d(x, \eta^{-1} \gamma^p \eta x) = \infty.$$

Assume the contrary, then there exists

- a constant R,

- a sequence of integers p_i going to infinity,

- a sequence of points x_i in K,

- a sequence of elements η_i of Γ,

such that $d(x_i, \eta_i^{-1} \gamma^{p_i} \eta_i x_i) \leqslant R$. It follows that

$$\forall i, (\eta_i^{-1} \gamma \eta_i)^{p_i} K_R \cap K_R \neq 0.$$

By the properness of the action of Γ, we conclude that the family $\{(\eta_i^{-1} \gamma \eta_i)^{p_i}\}$ is finite. But this contradicts the bounded depth root property.

We now choose the compact K such that $\Gamma.K = X$. It follows that

$$\lim_{p \to \infty} \inf_{x \in X} d(x, \gamma^p x) = \lim_{p \to \infty} \inf_{x \in K, \eta \in \Gamma} d(\eta x, \gamma^p \eta x) = \infty.$$

This is what we wanted to prove. Q.E.D.

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